

THE SUPERCOMPLEXIFICATIONS AND ODD BIHAMILTONIANS STRUCTURES

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Abstract. The general method of the complex supersymmetrization (supercomplexifications) of the soliton equations with the odd (bi) hamiltonian structure is established. New version of the supercomplexified Kadomtsev-Petvishvili hierarchy is given. The second odd Hamiltonian operator of the SUSY KdV equation generates the odd N=2 SUSY Virasoro - like algebra.

1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy of integrable soliton nonlinear evolution equations [1,2] is among the most important physically relevant integrable systems. Quite recently a new class of integrable systems motivated by the Toda field theory appeared both in the mathematical and in the physical literature .

On the other side the applications of the supersymmetry (SUSY) to the soliton theory provide us a possibility of the generalization of the integrable systems. The supersymmetric integrable equations [3-14] have drawn a lot of attention in recent years for a variety of reasons. In order to get a supersymmetric theory we have to add to a system of k bosonic equations kN fermions

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and $k(N-1)$ boson fields ($k=1,2,\dots,N=1,2,\dots$) in such a way that the final theory becomes SUSY invariant. Interestingly enough, the supersymmetrizations, leads to new effects (not present in the bosonic soliton's theory).

In this paper we describe a new method of the $N = 2$ supersymmetrization of the KP hierarchy, first time presented for (M)KdV equation in [13]. Our method contains the $N = 1$ supersymmetric KdV equation presented by Beekers [7] as a special case.

Here we would like to present new results on the supercomplexified method. We construct the supersymmetric Lax operator for our supercomplex KP hierarchy. This operator however does not generate the supersymmetric conserved currents defined in the whole superspace. We describe therefore special procedure which allows us to obtain whole chain of conserved currents for our supercomplex hierarchy.

The supercomplex Korteweg de Vries equation, considered as a special element of our KP hierarchy, constitute a bi-hamiltonian equation and is completely integrable. The second Hamiltonian operator of the supercomplexified KdV equation generates some odd $N = 2$ SUSY algebra in contrast to the even algebra SUSY $N = 2$ Virasoro algebra considered in [8-10].

This supercomplexification is a general method. In order to obtain the $N = 4$ supercomplex version of soliton's equation it is possible to use two nonequivalent methods. In both cases we obtain two different generalizations of $N = 4$ KdV equation. The model proposed in this paper, differs from the one considered in [9,12].

The idea of introducing an odd hamiltonian structure is not new. Leites noticed almost 20 years ago [15], that in the superspace one can consider both even and odd symplectic structures, with even and odd Poisson brackets respectively. The odd brackets (also known as antibrackets) have recently drawn some interest in the context of BRST formalism in the Lagrangian framework [16], in the supersymmetrical quantum mechanics [17], and in the classical mechanics [18,19].

2 Supercomplexification

We shall consider an $N = 2$ superspace with the space coordinates x and the Grassman coordinates $\theta_1, \theta_2, \theta_2\theta_1 = -\theta_1\theta_2, \theta_1^2 = \theta_2^2 = 0$.

The supersymmetric covariant derivatives are defined by

$$\partial = \frac{\partial}{\partial x}, \quad D_1 = \frac{\partial}{\partial \theta_1} + \theta_1 \partial, \quad D_2 = \frac{\partial}{\partial \theta_2} + \theta_2 \partial, \quad (1)$$

with the properties

$$D_1^2 = D_2^2 = \partial, \quad D_1 D_2 + D_2 D_1 = 0. \quad (2)$$

$$D_1^{-1} := D_1 \partial^{-1}, \quad D_2^{-1} := D_2 \partial^{-1}. \quad (3)$$

We define the integration over the $N = 2$ superspace to be

$$\int dX H(x, \theta_1, \theta_2) = \int dx d\theta_1 d\theta_2 H(x, \theta_1, \theta_2), \quad (4)$$

where Berezin's convention are assumed

$$\int d\theta_i \theta_j := \delta_{i,j}, \quad \int d\theta_i := 0. \quad (5)$$

We always assume that the components of the superfields and their derivatives vanish rapidly enough.

Let us now consider some classical evolution equation in the form

$$u_t = F(u, u_x, u_{xx}, \dots) = P * \text{grad } H(u, u_x, u_{xx}..), \quad (6)$$

where u is considered evolution function, P is the Poisson tensor, H is Hamiltonian of some dynamical system and grad denotes the functional gradient.

In order to get the $N = 2$ supersymmetric version of the equation (6) let us consider the following ansatz which in the next we will call as supercomplexification

$$u = (D_1 D_2 U) + i U_x, \quad (7)$$

where now U is some $N = 2$ superfield and i is imaginary quantity $i^2 = -1$.

Introducing this ansatz to equation (6) we obtain that superfield U evolves in the time as

$$U_t = G(U, U_x, U_{xx}, (D_1 U), (D_2 U), \dots) = D_2^{-1} D_1^{-1} \text{Re}(F) = \partial^{-1} \text{Im}(F), \quad (8)$$

where Re and Im denotes the real and imaginary part respectively. In order to guarantee validity of (8) we assume that F is chosen in such a way that

$$(D_1 \text{Re}(F)) = (D_2 \text{Im}(F)), \quad (9)$$

$$(D_2 \text{Re}(F)) = -(D_1 \text{Im}(F)), \quad (10)$$

always holds. It is a condition of solvability of our construction.

Examples.

Let us consider the famous KdV equation

$$u_t = -u_{xxx} - 6u_x u. \quad (11)$$

The supercomplex version of this equation is

$$U_t = -U_{xxx} - 6(D_1 D_2 U)U_x, \quad (12)$$

and takes the following form in the components

$$f_t = -f_{xxx} + 6g f_x, \quad (13)$$

$$g_t = \partial(-g_{xx} + 3g^2 - 3f_x^2), \quad (14)$$

$$\xi_{1t} = -\xi_{1xxx} + 6\xi_{1x}g + 6\xi_{2x}f_x, \quad (15)$$

$$\xi_{2t} = -\xi_{2xxx} - 6\xi_{1x}f_x + 6\xi_{2x}g, \quad (16)$$

where

$$U = f + \theta_1 \xi_1 + \theta_2 \xi_2 + \theta_2 \theta_1 g. \quad (17)$$

Notice that the bosonic part of the equations (13,14) does not contain any fermions fields. Hence in some sense this "supersymmetrization is without supersymmetry".

Now assuming that

$$U = \Pi + \theta_2 \Phi, \quad (18)$$

where Π and Φ are $N = 1$ superbosonic and superfermionic functions respectively, our supercomplex KdV equation (12) reduces to

$$\Pi_t = -\Pi_{xxx} - 6\Pi_x(D_1 \Phi), \quad (19)$$

$$\Phi_t = -\Phi_{xxx} + 6\Pi_x(D_1 \Pi_x) - 6\Phi_x(D_1 \Phi). \quad (20)$$

Notice that when $\Pi = 0$ then our equations (20) reduces to the $N = 1$ supersymmetric KdV equation considered by Beekers in [7].

Let us consider the supercomplexified version of the Nonlinear Schrödinger equation as a second example

$$f_t = -f_{xx} - 2g f^2, \quad (21)$$

$$g_t = g_{xx} + 2f g^2, \quad (22)$$

as the second example. Assuming that

$$f = (D_1 D_2 F) + iF_x, \quad (23)$$

$$g = (D_1 D_2 G) + iG_x, \quad (24)$$

we obtain

$$F_t = -F_{xx} - 2\partial^{-1}\left(2(D_1 D_2 G)(D_1 D_2 F)F_x + G_x(D_1 D_2 F)^2 - G_x F_x^2\right), \quad (25)$$

$$G_t = G_{xx} + 2\partial^{-1}\left(2(D_1 D_2 G)(D_1 D_2 F)G_x + F_x(D_1 D_2 G)^2 - F_x G_x^2\right). \quad (26)$$

3 N=2 Supercomplexified Kadomtsev-Petviashvili hierarchy

It well known that Korteweg de Vries equation as well as the Nonlinear Schrödinger equation are the members of the KP hierarchy. This hierarchy is defined by the following Lax operator

$$Lax := \partial + f\partial^{-1}g, \quad (27)$$

which generates the equations by the so called Lax pair representation

$$\frac{d}{dt}Lax = [Lax, (Lax)_+^n], \quad (28)$$

where n is an arbitrary natural number and $(+)$ denotes the projection onto purely superdifferential part of the pseudosuperdifferential element.

In order to define the Lax operator which generate our supercomplexified solitonic equations let us first define the supercomplexified algebra of pseudo-superdifferential elements Υ as a set of the following elements

$$\sum_{n=-\infty}^{+\infty} ((D_1 D_2 F_n) + F_{nx} \partial^{-1} D_1 D_2) \partial^n, \quad (29)$$

where F_n is an arbitrary $N = 2$ superfield. There are three different projection in this algebra

$$P_k(g) = \sum_{n=k}^{+\infty} ((D_1 D_2 F_n) + F_{nx} \partial^{-1} D_1 D_2) \partial^n, \quad (30)$$

where g is an arbitrary element belonging to Υ while k can take three values $k = 0, 1, 2$ only.

Now in order to obtain the supercomplexified version of the Lax operator (27) it is enough to replace f and g by the following substitution

$$f \Rightarrow (D_1 D_2 F) + F_x D_1 D_2 \partial^{-1}, \quad (31)$$

$$g \Rightarrow (D_1 D_2 G) + G_x D_1 D_2 \partial^{-1}, \quad (32)$$

in (27) and replace the projection $(+)$ by the projection P_1 defined by (30) with $k = 0$.

As an example let us consider the Lax pair representation for the supercomplexified N=2 KdV equation. The Lax operator for this case is

$$Lax := \partial + \partial^{-1}((D_1 D_2 F) + F_x D_1 D_2 \partial^{-1}), \quad (33)$$

and produces the supercomplexified KdV equation

$$\frac{d}{dt}Lax := [Lax, P_1(Lax^3)], \quad (34)$$

where

$$P_1(Lax^3) = \partial^3 + 3((D_1D_2F) + F_xD_1D_2\partial^{-1})\partial. \quad (35)$$

It is interesting to notice that this Lax operator does not produce any conserved currents defined in the whole $N = 2$ superspace. The traditional supersymmetric residual definition as the coefficients standing in $D_1D_2\partial^{-1}$ in Lax^n gives us that this coefficient after integration over whole superspace is zero.

4 Conserved currents for the supercomplexified KdV equation

It is easy to prove using the symbolic computer computations [20-21] that this equation does not possess any superbosonic currents. On the other side using the same technique it is easy to find superfermionic conserved currents. Let us explain the connections of these currents with the usual (classical) currents of the KdV equation. This connection is achieved in four steps.

First step. Let us supercomplexify an arbitrary conserved current of the classical KdV equation. By H_{nr} let us denote the real part of n -th conserved current after the supercomplexifications.

Second step. We compute the usual integral of H_{nr} . This can be denoted as

$$\int H_{nr}dx = K_n^0 + \int K_n^1dx. \quad (36)$$

Third step. Now we compute the supersymmetrical integral over first supersymmetrical variable from K_n^1 . It can be symbolically denoted as

$$\int (D_1K_n^1)dx = H_{1n} + (D_1 \int S_n^1dx). \quad (37)$$

Fourth step. Finally we compute the supersymmetrical integral over second supersymmetrical variable which is denoted as

$$\int (D_2S_n^1)dx = H_{2n}. \quad (38)$$

H_{1n} and H_{2n} are just the conserved superfermionic currents of the supercomplexified KdV equation.

Let us presents several superfermionic conserved currents of the supercomplexified KdV equation

$$H_{12} = \frac{1}{2} \int dx d\theta_1 d\theta_2 U(D_1 U_x), \quad (39)$$

$$H_{22} = \frac{1}{2} \int dx d\theta_1 d\theta_2 U(D_2 U_x), \quad (40)$$

$$H_{13} = \frac{1}{2} \int dx d\theta_1 d\theta_2 (-(D_2 U_{xxx})U + 4(D_2 U)(D_1 D_2 U)U_x), \quad (41)$$

$$H_{23} = \frac{1}{2} \int dx d\theta_1 d\theta_2 (-(D_1 U_{xxx})U + 4(D_1 U)(D_1 D_2 U)U_x). \quad (42)$$

Interestingly these currents does not contain the classical currents of the KdV equation in the bosonic or fermionic parts.

We have checked, using the symbolic computations [20-21] that this procedure could be aplyed to the whole supercomplexified KP hierarchy as well.

5 Odd Virasoro - like algebra

It is well known that Korteweg - de Vries equation constitute the so called bihamiltonian structure

$$u_t = \partial * grad\left(\frac{1}{2} \int dx (u_x u_x - 2u^3)\right) = Vir * grad\left(\frac{1}{2} \int dx u^2\right), \quad (43)$$

where

$$Vir := -\partial^3 - 2u\partial - 2\partial u. \quad (44)$$

The operator ∂ and Vir are two Poissons tensors which constitue with two different hamiltonians the so called bihamiltonian structure. Tensor Vir is connected with the Virasoro algebra, realized as the Poison bracket algebra, through the Fourier decomposition of the field u . Indeed introducing

$$u := \sum_{k=-\infty}^{+\infty} T_k exp(ikx) - \frac{1}{4}, \quad (45)$$

where T_k is some generator, we obtain that

$$\{u(x), u(y)\} = Vir * \delta(x - y), \quad (46)$$

are equivalent with the following Virasoro algebra.

$$\{T_n, T_m\} = (n + m)T_{n+m} + \delta_{n+m,0}(n^3 - n). \quad (47)$$

We succeeded to find the supercomplexified version of the bihamiltonian structure of the supercomplexified KdV equation. Let us define four operators

$$P_{21} := D_1 \partial - 2\partial^{-1}(D_1 D_2 U) D_1 - 2(D_1 D_2 U) \partial^{-1} D_1 + 2\partial^{-1} U_x D_2 + 2U_x \partial^{-1} D_2, \quad (48)$$

$$P_{22} := D_2 \partial - 2\partial^{-1}(D_1 D_2 U) D_2 - 2(D_1 D_2 U) \partial^{-1} D_2 - 2\partial^{-1} U_x D_1 - 2U_x \partial^{-1} D_1, \quad (49)$$

$$P_{11} := D_1^{-1}, \quad (50)$$

$$P_{12} := D_2^{-1}. \quad (51)$$

These operators generate the supercomplexified KdV equation (12)

$$U_t := P_{21} \frac{\delta H_{12}}{\delta U} = P_{22} \frac{\delta H_{22}}{\delta U} = P_{11} \frac{\delta H_{23}}{\delta U} = P_{12} \frac{\delta H_{13}}{\delta U} \quad (52)$$

where hamiltonians are defined by (36-39).

We can construct an $O(2)$ invariant bihamiltonian structure considering the linear combination of $P_{11} \pm P_{12}$, $P_{21} \pm P_{22}$ with $H_{12} \pm H_{22}$, $H_{14} \pm H_{24}$ respectively. These structures define the same supercomplexified KdV equation (12).

Notice that the operators P_{21} , P_{22} or $P_{21} \pm P_{22}$ play the same role as the Virasoro algebra in the usual KdV equation. There is a basic difference - our Hamiltonian operators generates the odd Poisson brackets in the odd super-space. In order to obtain the explicit realization of this algebra we connect the Hamiltonian operator $P_{21} - P_{22}$ with the Poisson bracket

$$\{U(x, \theta_1, \theta_2), U(y, \theta'_1, \theta'_2)\} = (P_{21} - P_{22})(\theta_1 - \theta'_1)(\theta_2 - \theta'_2)\delta(x - y), \quad (53)$$

where

$$U(x, \theta_1, \theta_2) = u_0 + \theta_1 \xi_1 + \theta_2 \xi_2 + \theta_2 \theta_1 u_1. \quad (54)$$

Introducing the Fourier decomposition of u_0, ξ_1, ξ_2, u_1

$$\xi_j := \sum_{s=-\infty}^{\infty} G_s^j e^{isx}, \quad j := 1, 2, \quad (55)$$

$$u_0 := i \sum_{s=-\infty}^{\infty} L_s e^{isx}, \quad u_1 := \sum_{s=-\infty}^{\infty} T_s e^{isx} - \frac{1}{4}, \quad (56)$$

in (64) we obtain

$$\{T_n, T_m\} = \{L_n, L_m\} = \{L_n, T_m\} = 0, \quad (57)$$

$$\{T_n, G_m^i\} = (n^2 - 1)\delta_{n+m,0} + (-1)^i 2 \frac{n-m}{m} T_{n+m} - 2 \frac{n^2 - m^2}{m} L_{n+m}, \quad (58)$$

$$\{G_n^i, L_m\} = (n - \frac{1}{n})\delta_{n+m,0} + 2 \frac{m-n}{nm} T_{n+m} + (-1)^i 2 \frac{m^2 - n^2}{nm} L_{n+m}, \quad (59)$$

$$\{G_n^i, G_m^i\} = (-1)^i 2 \frac{m^2 - n^2}{nm} (G_{n+m}^1 + G_{n+m}^2), \quad (60)$$

$$\{G_n^1, G_m^2\} = -2 \frac{m^2 - n^2}{nm} (G_{n+m}^1 - G_{n+m}^2). \quad (61)$$

These formulae define the closed algebra with the graded Jacobi identity [22]

$$\sum_{cycl(a,b,c)} (-1)^{[1+a][1+c]} \{a, \{b, c\}\} = 0, \quad (62)$$

where $[a]$ denotes the parity of a . It is the desired odd Virasoro - like algebra.

6 Supercomplexified N=4 KdV equation

It is possible to obtain in two different manner the supercomplexified N=4 version of the soliton equation. In the first manner we strictly speaking make duple supercomplexifications of the usual solitons equations while in the second method we supercomplexify the well know N=2 supersymmetric equations. Let us describe these methods.

In the first method we assume that the superfunction U which satisfy the equation (12) could be presented as

$$U := (D_3 D_4 W) + i W_x. \quad (63)$$

where now W is some $N = 4$ bosonic superfunction. Introducing this ansatz to (12) we obtain that W satisfy the dubly supercomplexified KdV equation

$$W_t := -W_{xxx} - 6\partial^{-1}((D^4 W)W_{xx} + (D_1 D_2 W)(D_3 D_4 W)), \quad (64)$$

where $D^4 = D_1 D_2 D_3 D_4$.

For the second method we consider the $N = 2$ supersymmetric KdV equation

$$\Phi_t := \partial(-\Phi_{xx} + 3\Phi(D_1 D_2 \Phi) + \frac{1}{2}(\alpha - 1)(D_1 D_2 \Phi^2) + \alpha \Phi^3). \quad (65)$$

where α can take three values $-2, 1, 4$ if we would like to consider the integrable extensions. We assume that the superfield Φ satisfy the $N = 2, \alpha = 4$ SUSY KdV equation (62), takes after the supercomplexification the following form

$$\Phi := (D_3 D_4 \Upsilon) + i \Upsilon_x, \quad (66)$$

where Υ is some $N = 4$ superboson field.

Substituting this form in (65) we obtain

$$\begin{aligned} \Upsilon_t := & -\Upsilon_{xxx} + 3(D_1 D_2 (\Upsilon_x (D_3 D_4 \Upsilon))) - 4\Upsilon_x^3 + \\ & (\Upsilon_x (D^4 \Upsilon) + (D_3 D_4 \Upsilon)(D_1 D_2 \Upsilon_x)) + 12(D_3 D_4 \Upsilon)^2 \Upsilon_x. \end{aligned} \quad (67)$$

It is the desired generalization of the $N = 4$ SUSY KdV equation, which is different from the one considered in [9,12].

Let us remark that using the similar methods described earlier it is possible to obtain the supercomplexification of the bihamiltonians structures, the Lax pair representations and the conserved currents. As an example let me present the Lax operator for the equations (64) and (67).

For equation (64) the Lax operator is

$$L := \partial^2 + (D^4 W) + (D_1 D_2 W_x) i_2 + (D_3 D_4 W_x) i_1 + W_{xx} i_1 i_2 \quad (68)$$

where $i_1 = \partial^{-1} D_1 D_2$, $i_2 = \partial^{-1} D_3 D_4$ and $i_1^2 = i_2^2 = -1$.

For equation (67) the Lax operator is

$$L := -\left(D_1 D_2 + (D_3 D_4 \Upsilon) + \Upsilon_x \partial^{-1} D_3 D_4\right)^2 \quad (69)$$

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